1. On the planet Ork, a new field phenomenon has been discovered that violates Maxwell’s equations. It is proposed that the Lagrangian density governing the new field behavior is given by

$$\mathcal{L} = \frac{1}{2} \left( \epsilon_0 \left( \dot{A} + \nabla \Phi \right)^2 - \frac{1}{\mu_0} \left( \nabla \times A \right) \cdot \nabla \Phi \right) + J \cdot A - q \Phi$$

where $\epsilon_0$ and $\mu_0$ are constants while the remaining field quantities $(A, \Phi, q, J)$ vary in space and time. The relevant fields of interests include

$$E = -\dot{A} - \nabla \Phi \text{ and } B = \nabla \times A$$

Show that the minimization of the Lagrangian density with respect to the vector potential $A$ violates the Maxwell equation given by:

$$\nabla \times \left( \frac{B}{\mu_0} \right) = J + \epsilon_0 \dot{E}$$

Show the new equation based on minimization of $\mathcal{L}$ with respect to $A$. 
2. Given the divergence relation

\[ \oint_{\Gamma_0} \phi^2 \nabla \phi \cdot dS = \int_{\Omega_0} \nabla \cdot (\phi^2 \nabla \phi) \, dV \]

If \( \phi \) must satisfy Laplace’s equation, show whether or not \( \phi \) uniquely satisfies Laplace’s equation based on the above integral equation for a finite boundary \( \Gamma_0 \) surrounding the domain \( \Omega_0 \). Consider what boundary conditions are required on this finite domain to ensure a unique solution.

Second, consider the following integral relation

\[ \oint_{\Gamma_0} (1 - \phi^2) \nabla \phi \cdot dS = \int_{\Omega_0} \nabla \cdot ((1 - \phi^2) \nabla \phi) \, dV \]

and determine the uniqueness on \( \phi \) for this integral equation if Laplace’s equation is to again be satisfied. In particular, what boundary conditions are required for uniqueness?
Question.

a. Consider a $n \times n$, real matrix $A$. A singular value decomposition (SVD) of the matrix is given by

$$A = U\Lambda V^T$$  \hspace{1cm} (1)

where $U$ and $V$ are $n \times n$ orthogonal matrices (i.e., $UU^T = I$ and $VV^T = I$) and $\Lambda$ is a diagonal matrix with all nonzero diagonal elements, i.e.,

$$\Lambda = \text{diag}\{\lambda_i\}, \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0.$$  \hspace{1cm} (2)

A polar decomposition is given by

$$A = RW$$  \hspace{1cm} (3)

where $R$ is a $n \times n$ symmetric matrix and $W$ is a $n \times n$ orthogonal matrix. What are $R$ and $W$ in terms of $U$, $\Lambda$ and $V$? That is, write the polar decomposition matrices in terms of the SVD matrices. (Hint: Look at $AA^T$.)

b. The matrix $A$ cannot only be expressed as (1), but also as

$$A = UEAFV^T$$  \hspace{1cm} (4)

where $E$ and $F$ are $n \times n$ orthogonal matrices, or equivalently

$$A = U\Lambda V^T$$  \hspace{1cm} (5)

where $U = UE$ and $V^T = FV^T$. Note that since the product of two orthogonal matrices is orthogonal, $U$ and $V$ are orthogonal and (5) is an alternative SVD. Describe orthogonal $E$ and $V$ such that

$$A = UAV^T = UEAFV^T.$$  \hspace{1cm} (6)
Spring 2012

Linear Algebra (JCO):

In applied science it is common to try to explain the behavior of a variable $b$ depending on certain variables $a_1, a_2, \ldots, a_n$ by hypothesizing a linear dependence

$$b = x_0 + x_1a_1 + x_2a_2 + \cdots + x_na_n$$

To derive the unknown coefficients $x_0, x_1, \ldots, x_n$ many observations are made. For each observation $i = 1, 2, \ldots, m$ numerical data

$$b_i; a_{i1}, a_{i2}, \ldots, a_{in}$$

are recorded. Thus, $b_i$ is the value of $b$ in the $i^{th}$ observation, and $a_{ij}$ is the value of $a_j$ in the $i^{th}$ experiment.

We then try to solve the equations

$$b_i = x_0 + x_1a_{i1} + x_2a_{i2} + \cdots + x_na_{in} \quad (i = 1, \ldots, m) \quad (1)$$

If the number of observations, $m$, is greater than the number of unknowns, $1 + n$, the linear system (1) usually has no solution. Nevertheless, one would like to find the best possible values for the unsolvable linear system. A common approach to define the 'best possible' solution is the least squares: we pick the numbers $x_0, x_1, \ldots, x_n$ to minimize the sum of the square of the difference:

$$\phi = \sum_{i=1}^{m} (b_i - x_0 - x_1a_{i1} - x_2a_{i2} - \cdots - x_na_{in})^2 \quad (2)$$

Consider the following:

$$A = \begin{pmatrix} 1 & a_{11} & \cdots & a_{1n} \\ 1 & a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$
a. Show that, in order to be an extremum of $\phi$, the vector $x$ that minimize $\phi$ satisfies $A^T Ax = A^T b$. You must clearly indicate all your steps to receive credit.

b. Suppose that $x$ is the solution to $A^T Ax = A^T b$. If $y$ is another vector, $y = x + z$, show that $x$ indeed minimizes $\phi$, that is, show that $\|Ay - b\|^2 \geq \|Ax - b\|^2$. You must clearly indicate all your steps to receive credit.

c. Illustrate the method of least squares by finding the best linear fit $b = x_0 + x_1 a_1 + x_2 a_2 + x_3 a_3$, if four observations give the data:

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by solving $A^T Ax = A^T b$. Report $A^T A$ and the vector $x$. You must clearly indicate all your steps to receive credit.
In biological systems a model exists that shows that a managed biological system population $x$ can achieve a stable and positive equilibria of $K$ if the harvesting (death) rate $m$ and the production (birth) rate $b$ the biologic resource are well managed by a net proportional growth rate $r$. That model is given by

$$\dot{x} = r(x)x$$

$$r(x) = b(x) - m(x) \approx c\left(1 - \frac{x}{K}\right)$$

where $c \left(1 - \frac{x}{K}\right)$ is a satisfactory approximation of $r(x)$.

Find the solution for $x(t)$ and show that the system converges to $x(\infty) = K$ for any initial condition $x(0) = x_0$. 
A system is described as
\[
\begin{align*}
x' &= (\delta + x) \sin y \\
y' &= 1 - x - \cos y + \epsilon \sin y
\end{align*}
\] (1)
where \(\delta\) and \(\epsilon\) are some real numbers.

1. Find all critical points.
2. Show if the system is almost linear in a neighborhood of each critical point.
3. Analyze the stability of the system around the critical points.
4. For which values of \(\delta\) and \(\epsilon\) is the system stable (near every critical point)?