1. Find the sum

$$\sum_{n=1}^{\infty} n (a^n + b^n)$$

where $a$ and $b$ are numbers between 0 and 1.

Derive your answer using standard results that are normally covered during or before the Calculus level.

Hint: Think of integration or differentiation, this being a calculus question.
If $A$ is 3 by 3 symmetric positive definite, then $Ax_i = \lambda_i x_i$ with positive eigenvalues and orthonormal eigenvectors $x_i$.

Let $y = c_1 x_1 + c_2 x_2 + c_3 x_3$.

a. Compute $y^T y$ and also $y^T Ay$ in terms of the $c$'s and the $\lambda$'s.

b. With the result of part (a), what values of $c$'s would make the ratio

$$\frac{y^T Ay}{y^T y}$$

as large as possible? You can assume $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Give the values of all three $c$'s.

c. Justify your answer in (b).
The ODE that results in the phase portrait displayed above has the form $\ddot{x} + \alpha x^2 + \beta = 0$.

1. Find the coefficients $\alpha$ and $\beta$ for the ODE given the directional field @$(1, 1) = [1, 0.75]$.
2. Given that $\alpha$ remains constant, what are the critical points at any arbitrary value of $\beta \neq 0$? List the type for each critical point. Also, plot the phase portrait for each structurally different ODE.
Question.
Consider the set of three vectors \( \{x_1, x_2, x_3\} \), where

\[
\begin{align*}
x_1 &= \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \\
x_2 &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \\
x_3 &= \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}.
\end{align*}
\]

a. Verify that these vectors are mutually orthogonal.

b. Find a vector \( x_4 \) with Euclidean norm (i.e., length) 1 such that \( \{x_1, x_2, x_3, x_4\} \)
   is a set of mutually orthogonal vectors.
1. Obtain the directional derivative of the second order tensor (i.e., matrix) $T = T_{ij} \hat{e}_i \hat{e}_j$ for $i, j = 1, 2, 3$ in the direction $n = \frac{1}{\sqrt{3}} \hat{e}_1 - \hat{e}_2 + \frac{1}{\sqrt{2}} \hat{e}_3$. Second, determine what is required to equate $\nabla \times T$ to $T \times \nabla$. 
ODE Qualifying Exam Question, Fall 2012

Solve the following ODE showing all your work.

\[ 2xe^{2t} + (1 + e^{2t}) \frac{dx}{dt} = 0 \]

\[ x(1) = -2 \]