1. The boundary of a two-dimensional area is described by

\[ F(x, y, t) = 0 \]

Derive the expression for the time derivative of the area in terms of function \( F \) and its derivatives. Justify all steps. Use very neat graphics to make your points.
Certain vector field gradient identities are very useful in decomposing field quantities for problems that utilize moving coordinates. In the following problem,

a) Prove the identity:

\[ \nabla \times (\mathbf{v} \times \mathbf{B}) = [\nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla] \mathbf{v} - [\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla] \mathbf{B} \]

where \( \mathbf{v} \) is velocity and \( \mathbf{B} \) is a vector field.

b) Using this relation, show that the time derivative for a moving coordinate frame over a fixed volume is

\[ \frac{d}{dt} \int \mathbf{B} dV = \int \left( \frac{\partial \mathbf{B}}{\partial t} + \mathbf{B} \cdot \nabla \mathbf{v} - \nabla \times (\mathbf{v} \times \mathbf{B}) - 2(\nabla \cdot \mathbf{v}) \mathbf{B} \right) dV + \int \hat{n} \cdot \mathbf{B} dS \]

where \( dV \) is a volume element, \( dS \) is the increment of the surface area, and \( \hat{n} \) is the unit normal on the area. Hint: \[ \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{B} \].
3. Answer the following questions:

(a) Construct a $4 \times 4$ matrix $A$ that has the following properties:
   
   i. $A$ is singular.
   ii. $A$ has no zero rows or columns.
   iii. $A$ has no row or column that is a multiple of another row or column.

(b) For the $A$ you constructed in (a) determine a $4 \times 1$ vector $y$ such that

   $$Ay = 0.$$  \hfill (1)

(c) Consider the equation

   $$Ax = b,$$ \hfill (2)

   where $A$ is the matrix you constructed in (a). Does equation (2) have a solution $x$ for any $(4 \times 1)$ $b$? Explain your answer and if your answer is no, describe the class of $b$'s for which a solution exist. You may choose to base your discussion on the following abstract representation of $A$:

   $$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix},$$ \hfill (3)

   where $a_i$ is $4 \times 1$ and denotes the $i^{th}$ column of $A$.

(d) Consider again the matrix $A$ you constructed in (a) and equation (2). Assume that there exists one solution $x_0$ such that

   $$Ax_0 = b.$$ \hfill (4)

   Are there additional solutions? If so, characterize at least some of them.

**IMPORTANT:** Do not use numbers in (c) and (d). In fact, you can do (c) and (d) even if you could not construct the matrix $A$ in (a). Simply use the properties that the matrix should have.
4. Consider three numbers, $x$, $y$ and $z$ such that they add to zero, $x + y + z = 0$. Let, $\theta$, be the angle between your vector $\vec{v} = (x, y, z)$ and the vector $\vec{w} = (z, x, y)$. Call $\alpha$ the cosine of that angle, $\alpha = \cos(\theta)$.

Consider the matrix

$$A = \begin{pmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{pmatrix}$$

which depends on a parameter $c$.

(a) (30%) Find all values of the parameter $c$ that make matrix $A$ positive definite. (Justify your steps.)

(b) (70%) Find the sign of $\vec{x}^T (\alpha A) \vec{x}$ if $c > 2$ (Justify your steps.)

Note: By definition a symmetric matrix is positive definite if the quadratic form $\vec{x}^T A \vec{x} > 0$ as long as $\vec{x} \neq 0$. A necessary and sufficient condition for a matrix to be positive definite is that all upper-left submatrices $A_k$ have a positive determinant. Here $A_k$ is the submatrix found by restricting the indices $i$ and $j$ in $a_{ij}$ to values no greater than $k$. So for a $3 \times 3$ matrix it requires:

$$a_{11} > 0 \quad \left| \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right| > 0 \quad \left| \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right| > 0$$
Preliminary Exam Question – ODE

You are given the system:

\[ \frac{dx}{dt} = Ax \]

Where \( x \in \mathbb{R}^{2 \times 1} \) and \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \).

If the system's initial position is sufficiently far from the origin, it will approach and follow a stable counterclockwise spiral with many loops about the origin. One of its many loops passes through the position \( x = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \) then proceeds to \( x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) in one complete loop around the origin.

1. Find the values of \( a, b, c \text{ and } d \) for this system.

2. Give the solution for this system for any initial condition \( x(0) = x_0 \).

Hint for #1: Figure out a suitable trace and determinant for the matrix \( A \) first, then it is not hard to write down a suitable set of possible answers for the entries of \( A \).
Question

Given the following system of differential equations:

\[
\dot{x}(t) = f(x(t), u(t)), \quad \text{where } x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}
\]

\[
\dot{x} = \begin{bmatrix} u \\ ux_1 - x_3 \\ x_2 - 2x_3 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0 \\ -3 \\ -2 \end{bmatrix}
\]

\[
y(t) = x_2 - 2x_3
\]

1) Linearize the system about the initial condition \( x_0 \).

2a) For the original system, with the initial condition \( x_0 \), if \( u \) is a constant \( (u = u_0) \), what values of \( u \) yield \( y(t) = 1 \).

2b) What is unusual about this system? Explain.